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Volumetric modification of the Hertz contact problem and its application to the multibody dynamics simulation †

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Abstract

A method of computational reduction of an elastic contact model for rigid bodies in frame of the Hertz contact model is considered. An algorithm to transform outer surfaces' geometric properties to the local contact coordinates system is described. It tracks permanently in time the surfaces of the bodies which are able to contact. An approach to compute the normal elastic force is represented. That one deals with the reduction to one transcendental scalar equation that includes the complete elliptic integrals of the first and second kinds. Simulation of the Hertz model was accelerated essentially due to use of the differential technique to compute the complete elliptic integrals and due to the replacement of the implicit transcendental equation by the differential one. Based on the Hertz contact problem classic solution, an invariant form for the force function which depends on the geometric properties of an intersection for undeformed rigid bodies' volumes, so-called volumetric model, is proposed then. The resulting force function reduced expression is supposed to be in use in cases when the classical contact theory hypotheses are broken. The expression derived has been applied to several cases of the elastic bodies contacting, and in particular back to the source Hertz model itself.

Keywords: Ball bearing model; Hertz contact model; Theorem of existence and uniqueness; Volumetric contact model

1. Introduction

It is known [1] that to compute the force of elastic bodies' interaction at a contact, several different approaches are applied: (a) the classical Hertz model [2], (b) model based on the polygonal approximation of the contacting surfaces [3] applied to cases of the surfaces of a complex shape, and (c) the volumetric model [1]. In our model we follow the classical Hertz approach, and the normal force computation method is the main topic of our analysis. To handle the surfaces at the contact we apply an approach defining the surfaces using equations of constraints. For definiteness and simplicity to simulate the tangent contact force one uses a regularized model of the Coulomb friction, [4]. This is sufficient enough to simulate the dynamics over time of the machine under simulation lifecycle. It may be that some additional complications for the friction model, e. g., account of the lubrication of any type, will be needed.

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2. Reduction in vicinity of contact

Keeping a frame of the formalism applied previously to simulate a unilateral constraint [4], consider its particular case corresponding to the mechanics of elastic contact interaction for two rigid bodies, identified hereafter as A and B. Their outer surfaces, see Fig. 1, being at contact supposed sufficiently regular.

How to compute the two opposing nearest points P_A and P_B of the surfaces to be tracked all over the simulation process? Applying the obvious notations

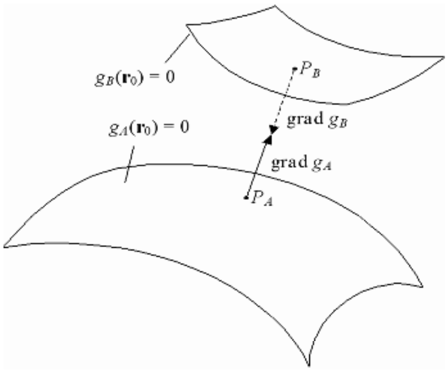

Fig. 1. Vicinity of the contact area.

we start here by reproducing the system of eight scalar equations:

$$\begin{aligned} \operatorname{grad} g_A\left(\mathbf{r}_{P_A}\right) &= \lambda \cdot \operatorname{grad} g_B\left(\mathbf{r}_{P_B}\right), \\ \mathbf{r}_{P_A} - \mathbf{r}_{P_B} &= \mu \cdot \operatorname{grad} g_B\left(\mathbf{r}_{P_B}\right), \\ g_A\left(\mathbf{r}_{P_B}\right) &= 0, \\ g_B\left(\mathbf{r}_{P_B}\right) &= 0. \end{aligned} \tag{1}$$

To define the coordinates x_{PA}, y_{PA}, z_{PA}, x_{PB}, y_{PB}, z_{PB} of the outer surfaces opposing points P_A, P_B, see Fig. 1. Here the coordinate vectors $\mathbf{r}_{PA} = (x_{PA}, y_{PA}, z_{PA})^T$, $\mathbf{r}_{PB} = (x_{PB}, y_{PB}, z_{PB})^T$ are defined with respect to (w. r. t.) the absolute coordinate frame $O_0x_0y_0z_0$ of reference (AF) usually connected to the multibody system base body B_0. Note the functions $g_A(\mathbf{r}_0) = g_A(\mathbf{r}_0, t)$, $g_B(\mathbf{r}_0) = g_B(\mathbf{r}_0, t)$ are really time dependent ones, and define outer surfaces' current spatial position of the bodies at a contact w. r. t. AF. The values λ, μ are auxiliary variables.

It turned out computationally that the most suitable approach to implement a system of algebraic equations like (1) is to replace it by the system of DAEs properly derived from (1). This can be done by introducing additional variables, being the time derivatives, and composing a differential subsystem of the form

$$\dot{\mathbf{r}}_{P_A} = \mathbf{u}_{P_A}, \quad \dot{\mathbf{r}}_{P_B} = \mathbf{u}_{P_B}, \quad \dot{\lambda} = \xi, \quad \dot{\mu} = \eta, \tag{2}$$

completed by the algebraic one

$$\begin{aligned} &[\omega_A, \operatorname{grad} g_A] + T_A \operatorname{Hess} f_A T_A^T\left(\mathbf{u}_{P_A} - \mathbf{v}_{P_A}\right) - \\ &\qquad \xi \operatorname{grad} g_B - \\ &\lambda\left([\omega_B, \operatorname{grad} g_B] + T_B \operatorname{Hess} f_B T_B^T\left(\mathbf{u}_{P_B} - \mathbf{v}_{P_B}\right)\right) = 0 \\ &\mathbf{u}_{P_A} - \mathbf{v}_{P_B} - \eta \operatorname{grad} g_B - \\ &\mu\left([\omega_B, \operatorname{grad} g_B] + T_B \operatorname{Hess} f_B T_B^T\left(\mathbf{u}_{P_B} - \mathbf{v}_{P_B}\right)\right) = 0 \\ &\left(\operatorname{grad} g_A, \mathbf{u}_{P_A}\right) - \left(\operatorname{grad} f_A, T_A^T \mathbf{v}_{P_A}\right) = 0 \\ &\left(\operatorname{grad} g_B, \mathbf{u}_{P_B}\right) - \left(\operatorname{grad} f_B, T_B^T \mathbf{v}_{P_B}\right) = 0 \end{aligned} \tag{3}$$

where the vectors $\mathbf{v}_{PA}$, $\mathbf{v}_{PB}$ are a velocities of bodies' physical points currently located at the geometric points P_A, P_B, and ω_A, ω_B are the angular velocities of the bodies. Matrices Hess f_A, Hess f_B are the Hesse ones of the functions f_A, f_B defining bodies' outer surfaces w. r. t. bodies' central principal coordinate systems. The functions f_A, f_B relate to the ones g_A, g_B according to the equations

$$g_\alpha(\mathbf{r}_0) = f_\alpha[T_\alpha^{\mathrm{T}}(\mathbf{r}_0 - \mathbf{r}_{O_\alpha})] \quad (\alpha = A, B),$$

where T_A, T_B are the orthogonal matrices defining current orientation of the bodies.

As usual for the Hertz approach we assume that the bodies A and B do not create any obstacles for their relative motion. If 3D-regions bounded by bodies' outer surfaces do not intersect, then the computer model of contact has to generate a zero wrench in the direction of each body. Simultaneously, it has to generate the radius vectors $\mathbf{r}_{PA}$, $\mathbf{r}_{PB}$ of opposite with each other points P_A, P_B.

Based on (1), note that the variable μ indicates the contact of the bodies A and B. Indeed, for definiteness suppose the outer surfaces in vicinities of the points P_A, P_A are such that vectors of gradients grad $g_A(\mathbf{r})$, grad $g_B(\mathbf{r})$ are directed outside of each body. Then we have the following cases at hand: (a) $\mu > 0$ means the contact is absent; (b) $\mu \le 0$: the contact takes place. If $\mu < 0$ then the bodies are supposed to penetrate each other, though they really begin to deform in the region of the contact. In the sequel we follow the simplest elastic contact model originating from Hertz [2]. Computational analysis will be performed only for the case of contacting (see Fig. 2). For simplicity and definiteness the surfaces are shown convex in Fig. 2, though it is not necessary at all in general for our implementation.

To represent the Hertz contact model in its classical form, first we have to construct an auxiliary bases in the vicinity of the contact. First base is composed by three unit vectors α, β, γ such that $\gamma = \mathbf{n}_A$, where $\mathbf{n}_A$ is the unit vector along the gradient grad $g_A(\mathbf{r})$ collinear to the z-axis in Fig. 2. As it was for the derivation of

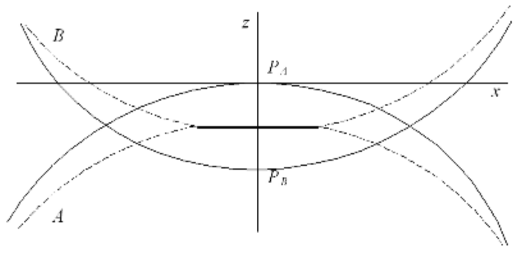

Fig. 2. Local coordinate system..

the opposing points, the most appropriate move to compute the proper base {α, β, γ} is to construct a relevant subsystem of DAEs. First, start with a differential equation for γ. It has the form

$$\dot{\boldsymbol{\gamma}} = |\operatorname{grad} g_A|^{-1}\left[\frac{d}{dt}\operatorname{grad} g_A - \left(\mathbf{n}_A, \frac{d}{dt}\operatorname{grad} g_A\right)\mathbf{n}_A\right].$$

After that we can write down the chain of equations $\boldsymbol{\Omega} = [\boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}}]$, $\dot{\boldsymbol{\alpha}} = [\boldsymbol{\Omega}, \boldsymbol{\alpha}]$, $\boldsymbol{\beta} = [\boldsymbol{\gamma}, \boldsymbol{\alpha}]$ defining successively the angular velocity **Ω** of the unit vector γ(*t*) rotation, the differential equation for the unit vector α, and the unit vector β completing the local base under construction. Actually the vector **Ω** is an angular velocity of the base triple {α, β, γ} w. r. t. *AF*.

Using the base {α, β, γ} built up above, it is easy to compose the matrix T = [α, β, γ] consisting of the columns composed themselves by the coordinates of the corresponding unit vectors. Actually, T is the transfer matrix between coordinates of *AF* and the current local base {α, β, γ}. This makes it possible to express an outer surface's equations in coordinates of the local system (*LF*) having an origin at the point P_A, (see Fig. 2). Under the general assumptions of the regularity for the bodies' outer surfaces, we can easily construct the procedure transforming the surface's equations permanently in time to the *LF* such that they can be resolved w. r. t. the variable z in the explicit form:

$$z = a'_\alpha x^2 + 2c'_\alpha xy + b'_\alpha y^2. \tag{4}$$

The further reduction comes to a transformation to canonical representation of the quadratic form

$$q(x, y) = ax^2 + 2cxy + by^2, \tag{5}$$

derived as a difference between the forms (4) such that $a = a'_B - a'_A$, $b = b'_B - b'_A$, $c = c'_B - c'_A$.

The transformation is implemented simply as a rotation about the z-axis of the system P_Axy to achieve the coefficient c vanishing. Finally, the function (5) has the form

$$q(x, y) = Px^2 + Qy^2 \tag{6}$$

with the additional condition $0 < P \le Q$.

3. The Hertz model

According to the known technique [5] to compute the total normal force at the contact, we have to solve a system of three transcendental equations (provided the coefficients P, Q from the representation (6) and depth of mutual penetration, so-called mutual approach, $h = |\mathbf{r}_{PB} - \mathbf{r}_{PA}|$ are have already been computed) of the form

$$\begin{aligned} \frac{FD}{\pi}\int_0^\infty \frac{d\xi}{\sqrt{(\alpha+\xi)(\beta+\xi)\xi}} &= h, \\ \frac{FD}{\pi}\int_0^\infty \frac{d\xi}{(\alpha+\xi)\sqrt{(\alpha+\xi)(\beta+\xi)\xi}} &= P, \\ \frac{FD}{\pi}\int_0^\infty \frac{d\xi}{(\beta+\xi)\sqrt{(\alpha+\xi)(\beta+\xi)\xi}} &= Q. \end{aligned} \tag{7}$$

The system (7) has three unknown variables: α, β, F, where the values α, β are the semi-major axes squared of the contact spot ellipse, and F is the total normal elastic force really distributed over the contact area. The parameter D summarizes elastic properties of the contacting bodies and depends on Poisson's ratios ν_A, ν_B and Young's moduli E_A, E_B.

Using the substitution $\xi \mapsto \eta$ ($\xi = \lambda\eta$) in elliptic integrals of (7) we can separate the last two equations of (7). Indeed, introducing new scaled unknown variables α', β' according to formulae $\alpha' = \alpha/\lambda$, $\beta' = \beta/\lambda$ we can deduce the two mentioned equations to the closed system of ones w. r. t. α', β' if the scaling factor λ satisfies the norming condition $FD\pi^{-1}\lambda^{-3/2} = 1$.

Furthermore, we can reduce this system of equations to the one-dimensional transcendental equation

$$\frac{1}{2}\mathsf{K}(c)\left(\frac{d\mathsf{K}(c)}{dc}\right)^{-1} - (1-c) = \frac{P}{Q} \tag{8}$$

w. r. t. the unknown value $c = k^2 = 1 - \beta'/\alpha'$, the elliptic integral modulus square. Here $\mathsf{K}(c)$ is the complete elliptic integral of the first kind. As one can clearly see, we interpret the complete elliptic integrals as a function of c using the work [6] as a pattern. Note that the inequality $\alpha' \geq \beta'$, which is equivalent to the condition $P \leq Q$, satisfies the above. As one can see here the value k actually has a geometric sense exactly of the contact spot ellipse eccentricity.

Once the solution of the Eq. (8) has been found we can obtain immediately the values

$$\alpha' = \left(\frac{4}{Q} \frac{d\mathsf{K}(c)}{dc} \right)^{2/3}, \quad \beta' = \alpha'(1-c).$$

Using the first equation of (7) and normalizing condition, we then find the value of the scaling factor λ, thus arriving at the Hertz problem solution: the normal force and the contact ellipse semi-major axes values $F = \pi D^{-1}\lambda^{3/2}$, $a = (\lambda\alpha')^{1/2}$, $b = (\lambda\beta')^{1/2}$.

Nevertheless, numeric implementation usually requires a further reduction of the model in a manner we already mentioned above: preferably use the differential equations (evidently to overcome the potential problems on the DAE system index reduction stage with a software at hand). To this end we have to remember the known ODEs concerning the complete elliptic integrals of the first $\mathsf{K}(c)$ and the second $\mathsf{E}(c)$ kind between one another [6]:

$$\frac{d\mathsf{K}}{dc} = \frac{1}{2}\frac{\mathsf{E} - \mathsf{K}(1-c)}{c(1-c)}, \quad \frac{d\mathsf{E}}{dc} = \frac{1}{2}\frac{\mathsf{E} - \mathsf{K}}{c}.$$

Furthermore, instead of (8) then we should use its differential version:

$$\left[3\left(\frac{d\mathsf{K}}{dc} \right)^2 - \mathsf{K}\frac{d^2\mathsf{K}}{dc^2} \right]\dot{c} = 2\left(\frac{d\mathsf{K}}{dc} \right)^2 \dot{C},$$

where $C = P/Q$ and

$$\frac{d^2\mathsf{K}}{dc^2} = \frac{(1-c)(2-3c)\mathsf{K} - (2-4c)\mathsf{E}}{4c^2(1-c)^2}.$$

In this way elliptic integrals become additional state variables, and simultaneously we have yet another way to compute elliptic integrals in dynamics (note: an exclusively fast and sufficiently accurate way).

Numerical behavior of the Hertz contact problem solution is well known [7]. But one other formal issue remains unresolved yet: whether the equation (8) has a unique solution c. It turns out the following analytic result takes place:

Theorem. *For the value $C = P/Q \in (0, 1]$ the equation (8) has exactly one solution on the set $c \in [0, 1)$.*

4. The volumetric model

Staying in the frame of the traditional Hertz model and taking into account that the expression for the normal force has the form $F_{\text{elast}} = -e(P, Q)h^{3/2}$, where while changing the value h the values P, Q do not change, we conclude the potential energy of elastic deformations is represented by the expression $U_{\text{elast}} = 0.4 \cdot e(P, Q)h^{5/2}$. On the other hand, using the volumetric approach one can try to represent the same potential energy as follows [8]: $U_{\text{elast}} = -f \ (b/a)V^{\nu}S^{\sigma}p^{\delta}$, where V is the volume of the bodies undeformed material intersected, S is the area of the intersection projection onto the xy-plane of the LF, p is the perimeter of that projection, a, b ($0 < a \leq b$) are the semimajor axes of the contact ellipse. It turns out that if $\nu = 2$, $\sigma = -7/4$, $\delta = 1/2$ then the function defined by the Vilke formula

$$V_{\text{elast}} = 0.357469 \ \frac{8}{15\pi^{1/4}(\theta_A + \theta_B)} \cdot \frac{V^2 p^{1/2}}{S^{7/4}}$$

differs from the exact Hertz U_{elast} by 0.5% of its value in a wide range of the contact ellipse shapes: surely for $b/a \in [0.1, 1]$. Here θ_α ($\alpha = A, B$) depends on bodies' material properties.

Since in case of the Hertz model the contact spot is the ellipse, then the values V, S, p can be computed explicitly. Then we can obtain the formula for an approximate value of the normal force at the contact (for each body in direction of h decreasing)

$$F_{\text{elast}} = -0.357469 \ \frac{2}{3(\theta_A + \theta_B)} \cdot \frac{\sqrt{\mathsf{E}(c_1)}}{P^{3/8}Q^{1/8}} h^{3/2}, \qquad (9)$$

where the elliptic integral modulus squared this time has the expression $c_1 = 1 - P/Q$.

Numeric experimental verification showed an application of the above expression for the normal force indeed causes a relative error near the value 0.5% for the contacting bodies configuration coordinates compared with the “exact” Hertz model over long time of simulation. Anyway, to estimate for example the fatigue processes in machines during the lifecycle simulation with the proper quality, it is sufficient enough to have an acceptable approximation for the contact forces because the Hertz model itself is surely the quasistatic approximation for the real processes of an elastic interaction.

The formula (9) is essentially simpler than computations in the Hertz model requiring the solution of the transcendental equation. The volumetric algorithm presented here is more reliable than the Hertz one; though, sometimes due to the differential techniques arranged for elliptic integrals the Hertz algorithm works even faster than the one above.

5. Example

The procedures described above to compute the normal force of an elastic interaction were implemented on Modelica language in the frame of a general approach to construct the objects of mechanical constraint [9]. Strictly speaking, in case of the compliant connection the constraint itself is absent. Instead, we have an elastic compliance implementing the Hertz contact model, though a general architecture of objects' interaction is conserved. Thus for future purposes the term "constraint" is retained.

Note that the implementation under consideration computes the normal force having besides the elastic Hertzian term the term of viscosity of the form $F_{\text{visc}} = -d(h)\dot{h}$, where h is the mutual approach. This latter term is assumed to arise due to the plasticity properties of the material the bodies are made of. It is fairly natural to consider the coefficient at dh/dt to depend upon h [10] since as the mutual approach increases from zero then the contact spot area also increases from zero. Therefore, it is quite natural for the plastic resistance to increase continuously from zero. A tangent force at the contact in our case for simplicity is implemented as a regularized model of the Coulomb friction [4]. Obviously, one can create here even far more complicated models for the tangent force at the contact.

An example of the ball bearing model is built up by using the architectural principle presented in [4]. For definiteness the bearing was equipped with eight balls. Each ball has two elastic contacts: one with the inner ring, and one with the outer one. In both cases when contacted, the ball simultaneously rolls over the surfaces of the toroidal tubes corresponding to the raceways of the inner and outer rings.

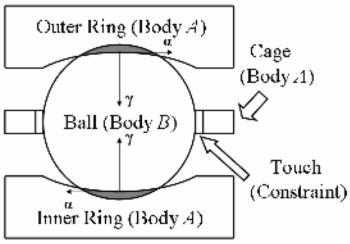

Fig. 3. Contacting ball and rings with the cage participation.

Here we describe in brief the specifications of the contact between the ball and one of the toroidal raceways. The ring is always assumed to be denoted as a body A in the contact object of the ball bearing model, while the ball is always denoted as B. All we need to complete the constraint specifications is to define the functions f_A, f_B. In our case we have

$$\begin{aligned} f_A(x,y,z) &= 4R_A^2\left(x^2+y^2\right) - \\ &\quad \left(x^2+y^2+z^2+R_A^2-r_A^2\right)^2, \\ f_B(x,y,z) &= x^2+y^2+z^2-R_B^2, \end{aligned}$$

where r_A is the toroidal pipe radius, R_A is the radius of the circle being an axis of that toroidal pipe, R_B is the ball radius. The geometry of the contacts inside the ball bearing can be seen in Fig. 3.

The outer and the inner rings are assumed to be connected rigidly with outer and inner shafts in our case, attached one with another by the bearing. In the example under consideration, the body connected with the outer ring rests w. r. t. AF while the body connected to the inner ring rotates uniformly about the z-axis of AF, both thus performing the prescribed motion (see the animation image in Fig. 4).

To verify the quality of the Hertz and volumetric models' implementation, we compared the vectors $\boldsymbol{\gamma}$ and $\mathbf{n}_A$ as functions of time. The computational experiments showed that their coordinates coincide with a very high accuracy. To compare two contact models under analysis, one can present for example the time dependences for the normal component of the contact force. The Hertz and the volumetric cases corresponding to the same contact object inside the ball bearing model showed a high degree of a coincidence.

And yet a final remark: to make the simulation even faster, by at least twice, one can apply the simplified expression of the form $F_{\text{elast}} = -eh^{3/2}$ with the constant coefficient e for the normal elastic force at the contact [11]. But it is possible only if the geomet-

Fig. 4. Animation of the ball bearing model.

ric properties (curvatures etc.) do not change while the model is simulated.

6. Conclusions

One can complete the results presented above by splitting the issues to the several main remarks influencing the potential directions of future work:

(1) According to an experience accumulated while developing the models simulating the multibody dynamics one can resume the usefulness of an approach when the differential formulations proper applied are more preferable than usual algebraic or even transcendental ones. Mostly it is because the DAE solvers work better if the source system of the model equations is best prepared to be differentiated to reduce the DAE index.
(2) In particular, it turned out that the introduction of the ODEs system components for outer surfaces' tracking for the elastic bodies contact problem conserves the accuracy and simultaneously improves the reliability of the models.
(3) Implementation of the complete elliptic integrals using the ODEs subsystem also was useful: the models became more reliable and fast. For instance, the Hertz algorithm, improved as described above, turned out to be even faster than the volumetric one in case of the contact area not very far from the circular case.
(4) The volumetric algorithm is more reliable and suitable for a wide range of the contact area eccentricities, simultaneously providing an accuracy of 0.5% with respect to the Hertz-point algorithm.
(5) In addition, the volumetric algorithm makes it possible to implement the contact force computation in a variety of cases far from the Hertz model for non-elliptic contact spots.

One can outline a broad line of the future work directions, such as the development of more complicated tangent forces models, account for the lubrication at the contact, applications to different types of appliances with the rotary motions, or to problems essentially including the effects of friction when contacting.

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